Shear Strength of Soil

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Principal stress space

\[ \sigma_a \]

\[ \sigma_b \]

\[ \sigma_c \]
Rendulic Plot

\[ \bar{\sigma}_a \]

\[ \bar{\sigma}_r \]

\[ \sqrt{2} \bar{\sigma}_r \]
Rendulic Plot

Axial Stress $\sigma_a$, Radial Stress $\sqrt{2} \, \sigma_r$

We can plot stress paths of compression and extension tests without having them cross over one another.
Redulic Diagram

1. S-test: compression by increase of axial stress
2. $R$-test: compression by increase of axial stress

3. S-test: compression by decrease confining stress
4. Elastic material: on octahedral plane
\[ \bar{\sigma}_1 = \bar{\sigma}_3 \left[ \frac{1 + \sin \bar{\phi}}{1 - \sin \bar{\phi}} \right] + 2 \bar{c} \frac{\cos \bar{\phi}}{1 - \sin \bar{\phi}} \]

\[ \bar{\sigma}_1 = \bar{\sigma}_3 \sqrt{2} \left[ \frac{1}{\sqrt{2}} \cdot \frac{1 + \sin \bar{\phi}}{1 - \sin \bar{\phi}} \right] + 2 \bar{c} \frac{\cos \bar{\phi}}{1 - \sin \bar{\phi}} \]

\[ \bar{\sigma}_a = \bar{\sigma}_r \sqrt{2} \left[ \frac{1}{\sqrt{2}} \cdot \frac{1 + \sin \bar{\phi}}{1 - \sin \bar{\phi}} \right] + 2 \bar{c} \frac{\cos \bar{\phi}}{1 - \sin \bar{\phi}} \]

\[ \tan \psi \]

\[ d \]
Redulic Diagram

2. S-test: extension by increase of radial stress
3. $\overline{R}$-test: extension by decrease of axial stress

1. S-test: extension by decrease axial stress
4. Elastic material: on octahedral plane
\[
\bar{\sigma}_r = \bar{\sigma}_a \left[ \frac{1 + \sin \phi}{1 - \sin \phi} \right] + 2\bar{c} \frac{\cos \phi}{1 - \sin \phi}
\]

\[
\bar{\sigma}_a = \bar{\sigma}_r \left[ \frac{1 + \sin \phi}{1 - \sin \phi} \right] - 2\bar{c} \frac{\cos \phi}{1 - \sin \phi}
\]

\[
\bar{\sigma}_a = \bar{\sigma}_r \sqrt{2} \left[ \frac{1}{\sqrt{2}} \cdot \frac{1 + \sin \phi}{1 - \sin \phi} \right] - 2\bar{c} \frac{\cos \phi}{1 - \sin \phi}
\]

\[
tan \psi,
\]

\[
d
\]
e.g. \( \bar{c} = 0 \quad \bar{\phi} = 30^\circ \)

For compression

\[
\tan \psi = \frac{1}{\sqrt{2}} \cdot \frac{1 + \sin \bar{\phi}}{1 - \sin \bar{\phi}} \quad \psi = 65^\circ
\]

For extension

\[
\tan \psi = \frac{1}{\sqrt{2}} \cdot \frac{1 - \sin \bar{\phi}}{1 + \sin \bar{\phi}} \quad \psi = 13^\circ
\]

For space diagonal

\[
\angle \tan^{-1} \left( \frac{1}{\sqrt{2}} \right) = 35^\circ
\]
e.g. \( \bar{c} = 0 \), \( \bar{\phi} = 30^\circ \)
### Henkel (1967)

<table>
<thead>
<tr>
<th>Compression Tests</th>
<th>Tests</th>
<th>$\bar{\phi}$</th>
<th>Extension Tests</th>
<th>Tests</th>
<th>$\bar{\phi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Undrained</td>
<td>23°</td>
<td></td>
<td>Undrained</td>
<td>22.5°</td>
<td></td>
</tr>
<tr>
<td>Drained, $\sigma_a \uparrow$</td>
<td>22°</td>
<td></td>
<td>Drained, $\sigma_a \downarrow$</td>
<td>22.5°</td>
<td></td>
</tr>
<tr>
<td>Drained, $\sigma_r \downarrow$</td>
<td>22°</td>
<td></td>
<td>Drained, $\sigma_r \uparrow$</td>
<td>21.5°</td>
<td></td>
</tr>
<tr>
<td>Drained, $\sigma_{\text{mean}} = \text{const.}$</td>
<td>22°</td>
<td></td>
<td>Drained, $\sigma_{\text{mean}} = \text{const.}$</td>
<td>22°</td>
<td></td>
</tr>
</tbody>
</table>
Monitor the volume of water flowed into or out of the specimen during CD tests or trace effective stress path of CU tests

Contour of constant water content (stress path of CU test)

\[ \overline{\sigma}_a = \overline{\sigma}_r \]

\[ w = 17\% \]

\[ w = 20\% \]
For $c=0$

\[ \sigma_a = \sigma_b = \sigma_c \]

Octahedral plane

\[
\overline{\sigma}_m = \frac{1}{3}(\overline{\sigma}_a + \overline{\sigma}_b + \overline{\sigma}_c) = \overline{\sigma}_{oct}
\]
Stress is constant for any point on the octahedral plane

\[ \bar{\sigma}_{oct} = \text{const.} \]

\[ r = \frac{1}{\sqrt{3}} \sqrt{(\bar{\sigma}_a - \bar{\sigma}_b)^2 + (\bar{\sigma}_b - \bar{\sigma}_c)^2 + (\bar{\sigma}_c - \bar{\sigma}_a)^2} \]

\[ r = \frac{1}{\sqrt{3}} \tau_{oct} \]

\[ \tau_{oct} = \text{octahedral shear stress} \]
\[ = \text{maximum shear stress on the octahedral plane} \]

\[ \tau_{oct} = \frac{1}{3} \sqrt{(\bar{\sigma}_a - \bar{\sigma}_b)^2 + (\bar{\sigma}_b - \bar{\sigma}_c)^2 + (\bar{\sigma}_c - \bar{\sigma}_a)^2} \]
Failure Criteria

- Mohr-Coulomb
- Tresca (Extended Tresca)
- von Mises
Mohr-Coulomb

For \( \sigma_1 > \sigma_2 > \sigma_3 \)

Independent of \( \sigma_2 \)

\[
(\sigma_1 - \sigma_3) = (\sigma_1 + \sigma_3) \sin \phi
\]

\[
\bar{\sigma}_{oct} = \frac{1}{3}(\bar{\sigma}_a + \bar{\sigma}_b + \bar{\sigma}_c) = \text{const.}
\]

\[
[(\sigma_1 - \sigma_2)^2 - (\sigma_1 + \sigma_2)^2 \sin \phi] \times
\]

\[
[(\sigma_2 - \sigma_3)^2 - (\sigma_2 + \sigma_3)^2 \sin \phi] \times
\]

\[
[(\sigma_3 - \sigma_1)^2 - (\sigma_3 + \sigma_1)^2 \sin \phi] = 0
\]
For $c=0$ 

If $c$ is not 0, but $\phi$ is, the shape would be a hexagonal column.
Tresca

\[(\sigma_1 - \sigma_3) = \alpha \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \alpha \sigma_{oct}\]

Often used for \(c=0\) and \(\phi=0\)

\(\alpha\) is the Tresca parameter equivalent to \(\phi\)

Extended Tresca for \(\phi > 0\)

\[
\begin{align*}
|\sigma_a - \sigma_b| & \leq \alpha \cdot \sigma_{oct} \\
|\sigma_b - \sigma_c| & \leq \alpha \cdot \sigma_{oct} \\
|\sigma_c - \sigma_a| & \leq \alpha \cdot \sigma_{oct}
\end{align*}
\]
Von Mises ($\phi = 0$)

$$\left(\sigma_a - \sigma_b\right)^2 + \left(\sigma_b - \sigma_c\right)^2 + \left(\sigma_c - \sigma_a\right)^2 = 2\alpha^2 \left(\frac{\sigma_a + \sigma_b + \sigma_c}{3}\right)^2$$

Extended Tresca for $\phi > 0$

- $c = 0 \rightarrow$ cone
- $\phi = 0 \rightarrow$ cylinder

$$9\tau_{oct} = 2\alpha^2 \sigma_{oct}^2 = \text{const.}$$

$\rightarrow$ Circle
If $\sigma_b = \sigma_c$

$$(\sigma_a - \sigma_b)^2 = \alpha^2 \left( \frac{\sigma_a + 2\sigma_b}{3} \right)^2$$
Bishop
\[ b = \frac{\bar{\sigma}_2 - \bar{\sigma}_3}{\bar{\sigma}_1 - \bar{\sigma}_3} \]

Mohr-Coulomb
\[ \frac{\bar{\sigma}_1 - \bar{\sigma}_3}{\bar{\sigma}_1 + \bar{\sigma}_3} = \sin \phi \]

Extended Tresca
\[ \frac{\bar{\sigma}_1 - \bar{\sigma}_3}{\bar{\sigma}_1 + \bar{\sigma}_3} = \frac{1}{\left( \frac{1}{3} + \frac{2}{\alpha} - \frac{2b}{3} \right)} \]

Extended von Mises
\[ \frac{\bar{\sigma}_1 - \bar{\sigma}_3}{\bar{\sigma}_1 + \bar{\sigma}_3} = \frac{1}{\left( \frac{1}{3} + \frac{2}{\alpha} \sqrt{1 - b + b^2} - \frac{2b}{3} \right)} \]
\[ \phi_f = \sin^{-1} \left( \frac{\overline{\sigma}_1 - \overline{\sigma}_3}{\overline{\sigma}_1 + \overline{\sigma}_3} \right) \]

Extended von Mises

Extended Tresca

M-C T.E.

M-C T.C.
Triaxial compression tests:

- Mohr-Coulomb criteria underestimate strength in plane strain
- Von Mises criteria overestimate strength in plane strain
Krey (1927)

Consolidated-Undrained Direct Shear Tests (Actually, it’s difficult to control drainage in DS tests)

Consolidate the specimen and then quickly reduce $\sigma$ and then shear it

\[ \tau_f = c + \sigma'_m \tan \phi' \]

Stress path

\[ \sigma'_m \]

\[ \sigma_{\text{max}} \]
Tiedemann (1933)

\[ \tau_f = \sigma_m \tan \phi_c + \sigma \tan \phi_r \]

The maximum past pressure the soil was consolidated

\[ c = f(\sigma_m) \]

If truly undrained, \( \phi_r \) should be zero. \( \Rightarrow \)

\[ \tau_f = \sigma_m \tan \phi_c \]
Gibson (1953) Ph.D Dissertation

\[ \phi_r \]

Bentonite, P.I. = 530

4°

P.I.
Hvorslev

Drained direct shear tests

True cohesion $\bar{c}_e = K \bar{\sigma}_e$

$\rightarrow$ It is a function of the void ratio

Virgin consolidation curve
\( \bar{\phi}_e \) True friction

\[ \tau_f = K \bar{\sigma}_e + \bar{\sigma} \tan \bar{\phi}_e \]

\[ \frac{\tau_f}{\bar{\sigma}_e} = K + \frac{\bar{\sigma}}{\bar{\sigma}_e} \tan \bar{\phi}_e \]

This translates into a SET of envelopes \( \rightarrow \) one envelope for each \( e \) or \( \frac{\bar{\sigma}}{\bar{\sigma}_e} \) because \( \bar{c}_e = K \bar{\sigma}_e \)
He took the soil out of shear zone and study the distribution of stress carefully

<table>
<thead>
<tr>
<th>Clay</th>
<th>( w_L )</th>
<th>( w_{Pl} )</th>
<th>&lt; 2 ( \mu )</th>
<th>( K )</th>
<th>( \bar{\phi}_e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wiener Tegel</td>
<td>47</td>
<td>25</td>
<td>23%</td>
<td>0.1</td>
<td>17.5°</td>
</tr>
<tr>
<td>Kleil Belt Ton</td>
<td>127</td>
<td>91</td>
<td>77%</td>
<td>0.145</td>
<td>10.0°</td>
</tr>
</tbody>
</table>

As \( w_L, w_{Pl}, < 2\mu \uparrow \rightarrow K \uparrow \) and \( \bar{\phi}_e \downarrow \)
Bjerrum (1954) Ph.D Dissertation

- Basically unsuccessful in determining the Hvorslev coefficients

\[
\tau_f = \bar{c}_0 + K \bar{\sigma}_e + \bar{\sigma} \tan \bar{\phi}_e
\]

\[
\bar{c}_e = \bar{c}_0 + K \bar{\sigma}_e
\]

Intercept cohesion
● Tests on specimens of the same void ratio tends to have same strength →
● Hard to determine different $\tau$ →
● Hard to determine $\bar{c}_0$ and $K\overline{\sigma}_e$
Test by consolidating the specimens to different $\sigma_{max}$, unload to the same $e$ and shear

During shear: no drainage $e = \text{const.}$

Critical confining stress

Virgin consolidation curve
Tests on specimens of the same void ratio tends to have same strength!

\[ \sigma_1 - \sigma_3 \]

same \( \sigma_1 - \sigma_3 \)

S.P.T.

Hvorslev envelope

No change in effective stress for the specimen sheared at critical void ratio
Critical State Soil Mechanics

\[ p = \frac{I_1}{3} \]

\( I_1 = \) first stress invariant
\[ = \sigma_1 + \sigma_2 + \sigma_3 \]

\[ p = \frac{1}{3}(\overline{\sigma}_1 + \overline{\sigma}_2 + \overline{\sigma}_3) = \overline{\sigma}_{oct} \]

For triaxial compression:

\[ \sigma_2 = \sigma_3 \]
\[ p = \frac{1}{3}(\overline{\sigma}_1 + 2\overline{\sigma}_3) \]

MIT people:
\[ p = \frac{1}{2}(\overline{\sigma}_1 + \overline{\sigma}_3) \]
\[ q = \sqrt{3 \times J_2} \]

\( J_2 = \) deviatoric component of the second stress invariant

\[ J_2 = -\sigma_1 \sigma_2 - \sigma_2 \sigma_3 - \sigma_3 \sigma_1 + 3\sigma_{oct}^2 \]

\[ J_2 = \frac{3}{2} \tau_{oct}^2 \]

\[ q = \frac{3}{\sqrt{2}} \tau_{oct} \]

\[ \tau_{oct} = \frac{1}{3} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \]
For triaxial compression:

\[ \sigma_2 = \sigma_3 \]

\[ \tau_{\text{oct}} = \frac{1}{3} \sqrt{2(\sigma_1 - \sigma_3)^2} \]

\[ q = (\sigma_1 - \sigma_3) \]

\[ p = \frac{1}{3} (\bar{\sigma}_1 + 2\bar{\sigma}_3) \]

\[ q = (\sigma_1 - \sigma_3) \]
For $c=0$

Space diagonal
\[
y = \frac{1}{\sqrt{3}} (\bar{\sigma}_1 + \bar{\sigma}_2 + \bar{\sigma}_3) = \sqrt{3} \bar{\sigma}_{oct} = \sqrt{3} p
\]

\[
z = \sqrt{3} \tau_{oct} = \sqrt{\frac{2}{3}} q
\]
\[ \sqrt{\frac{2}{3}} q \]

Tension zone

\[ \sigma_3 = 0 \]

\[ \sqrt{2} \]

\[ \sqrt{3} p \]
Tension zone

State path: $p, q, e$ (or other variables) path
N.C. Clay Undrained Loading

Envelope of NC clay = Projection of critical void ratio (c.v.r.) line on \( p, q \) plane

Hvorslev’s envelopes

S.P.T.

NC stress path
Virgin isotropic consolidation curve
Overconsolidate Clay – Undrained loading

Increase OCR

Virgin consolidation curve
same void ratio tends to have same strength → common point (at large strain)

S.P.T.

same $\sigma_1 - \sigma_3$

projection of critical void ratio (c.v.r.) line on $p$, $q$ plane

points on critical void ratio (c.v.r.) line

$\bar{\sigma}_3$ $\bar{\sigma}_{3f}$

No change in effective stress for the specimen sheared at critical void ratio
Drained tests

- $p, q, e$ are changing throughout the test
Roscoe, S, and Wroth

- They did not distinguish between $(\sigma_1 - \sigma_3)_{\text{max}}$ and $(\bar{\sigma}_1/\bar{\sigma}_3)_{\text{max}}$
- Useful in pile driving where the strain is very large
- As a building block for developing other stress-strain models
- They adopted von Mises criteria in order to get from 2-D to 3-D
Line of critical overconsolidation ratio

- A line on the $q = 0$ plane that marks the boundary of soil tends to dilate or contract.
Drained test: no change in $e$

Critical OCR line

Start from the left, dilate, $e \uparrow$
Start from the right, contract, $e \downarrow$

Virgin consolidation curve

C.V.R. line

$e (w)$

$\bar{p}$

Dry

Wet
- **Dry of critical**: tends to dilate during shear, heavily overconsolidated clay (on the left of the critical OCR line)
- **Wet of critical**: tends to contract during shear, normally consolidated or slightly overconsolidated clay (on the right of the critical OCR line)
Undrained Tests

The only starting point with no pore water pressure change is on the Critical OCR line.

\[ \Delta u = 0 \]
On C.V.R.

Critical OCR line
The only starting point with no pore water pressure change
Critical OCR line

C.V.R. Line

Projection of stress path on the 3:1 plane

Drained path

Undrained path

$e$

$\bar{p}_1$ $\bar{p}_2$ $\bar{p}_3$ $\bar{p}_4$ $\bar{p}$
Critical OCR line (R, S, & W)

- Depends on
  - The nature of the tests
  - Soil properties
- May not be parallel to the virgin consolidation curve
Partly Saturated Soils

\[ \tau_Q = c_Q + \sigma \tan \phi_Q \]

For partly saturated soils volume changes in Q tests
$S_r$ different $\rightarrow$ effective stress is different
S envelopes – after saturation

Essentially one envelope regardless of $S_r$ at compaction
Capillary pressure

- Whenever there is a curved air-water interface, there’s must be pressure drop
Capillary and the Capillary Fringe

- Capillary: water molecules at the water table subject to an upward attraction due to surface tension of the air-water interface and the molecular attraction of the liquid and the solid phases.
Tension

- If fluid pressures are measured above the water table, they will be found to be negative with respect to local atmospheric pressure.
Air Pressure

- If air in the pores is connected, the air pressure is equal to the local atmospheric pressure
- If air in the pores is not connected and is in the forms of air bubbles such as in the capillary fringe, the air pressure is equal to the pressure of water, which is negative.
Capillary rise in a tube

\[ \sigma \cos \lambda = T_s \]

\[ \lambda = 0 \]
\[ R = r \]

\[ h_c = \frac{2\sigma \cos \lambda}{\rho_w g R} \]

\[ u_c = h_c \gamma_w = 2 \frac{T_s}{\gamma_w} \]
Fig. 6.2  IDEALIZED pore diameter in a sediment with cubic packing. The equivalent capillary tube has a radius of 0.2 the diameter of the grains.
The diagram shows a water column with atmospheric pressure $P_{atm}$ acting on the top. The pressure head $\psi_h$ is constant and equal to 0. The diagram includes a scale for $z$ (m) with marks at 0.1, 0.2, 0.3, and 0.4. The pressure head $\psi_p$ and $\psi_g$ are indicated with lines intersecting the $z$ axis. The horizontal axis represents $y$ (J/kg) with values ranging from -4 to 4, and the vertical axis represents $r_1y$ (kJ/cu.m = kPa) with values ranging from -4 to 4. The bottom axis represents $y/g$ (J/N = m) with values ranging from -0.4 to 0.4.
\[ \psi_g = gz \]
$\psi_g = g z$
# Height of Capillary Rise

<table>
<thead>
<tr>
<th>Sediments</th>
<th>Grain Diameter (cm)</th>
<th>Pore Radius (cm)</th>
<th>Capillary Rise (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fine silt</td>
<td>0.0008</td>
<td>0.0002</td>
<td>750</td>
</tr>
<tr>
<td>Coarse silt</td>
<td>0.0025</td>
<td>0.0005</td>
<td>300</td>
</tr>
<tr>
<td>Very fine sand</td>
<td>0.0075</td>
<td>0.0015</td>
<td>100</td>
</tr>
<tr>
<td>Fine sand</td>
<td>0.0150</td>
<td>0.003</td>
<td>50</td>
</tr>
<tr>
<td>Medium sand</td>
<td>0.03</td>
<td>0.006</td>
<td>25</td>
</tr>
<tr>
<td>Coarse sand</td>
<td>0.05</td>
<td>0.010</td>
<td>15</td>
</tr>
<tr>
<td>Very coarse sand</td>
<td>0.20</td>
<td>0.040</td>
<td>4</td>
</tr>
<tr>
<td>Fine gravel</td>
<td>0.50</td>
<td>0.100</td>
<td>1.5</td>
</tr>
</tbody>
</table>
Water Potential

- The potential energy, or force potential of ground water consists of two parts: elevation and pressure (velocity related kinetic energy is neglected)
Fluid pressures in the vadose zone are negative, owing to tension of the soil-surface-water contact.

The negative pressure head is measured in the field with a tensiometer.
Soil Surface

Δz_1

Δz_2

Suction – Calibrate Before Use
Suction

- Matric suction
  - Elevation head
  - Pressure head
- Osmotic suction
Head (Water Potential)

- Gravity potential, $Z$
  - Elevation head
- Moisture potential, $\psi$
  - Suction head
  - Can be several orders of magnitude greater than the gravity potential
  - 1 bar $\approx$ 10 m of water column
Soil Water Characteristics

- Defines the relationship between water content and water potential (suction)
SWCC of a Coarse Sand
Soil Water Characteristic Curve
SWCC of various soils
Absorption and desorption characteristics with primary scanning curves.

Figure 3.14  Absorption and desorption characteristics with primary scanning curves.

Absorption and desorption characteristics with primary scanning curves
Water content, $\theta$

Pressure head (cm)

Poorly sorted

Well sorted

Entry pressure $h_b$
Absorption and desorption characteristics with primary scanning curves
Hysteresis

- Ink bottle effect
- Trapping of air
- Advancing and receding contact angle
Determination of SWCC

- Laboratory
  - Pressure plate apparatus
  - Filter paper
  - Thermocouple Psychrometer
  - Centrifuge
- Field
  - Tensiometer
  - Water content measurement
Effluent port that can vary elevation

$\psi_g = gz$

Ceramic plate

soil

water

heads (m)

$z (m)$

-1.6 -1.2 -0.8 -0.4 0 0.4

-1.2 -0.8 -0.4 -0.2 0

-1.6 -1.2 -0.8 -0.4 0 0.4 heads (m)
Pychrometers

- Measure relative humidity in close proximity to air-water interface

\[ h_c \left( \frac{u_c}{\gamma_w} \right) = -\frac{RT}{Mg} \ln\left( \frac{H}{100\%} \right) \]

\( H \) = relative humidity
\( R \) = the gas constant
\( T \) = absolute temp.
\( M \) = molecular wt. of water vapor
\( g \) = acceleration due to gravity
<table>
<thead>
<tr>
<th>Typical values of $u_c$ (psi)</th>
<th>$H$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.42</td>
<td>99.99</td>
</tr>
<tr>
<td>14.2</td>
<td>99.92</td>
</tr>
<tr>
<td>142.0</td>
<td>99.28</td>
</tr>
<tr>
<td>1420.0</td>
<td>93.0</td>
</tr>
<tr>
<td>14200.0</td>
<td>48.43</td>
</tr>
<tr>
<td>Amount of NaCl in water (g/L)</td>
<td>Osmotic suction at 20°C (bars)</td>
</tr>
<tr>
<td>------------------------------</td>
<td>--------------------------------</td>
</tr>
<tr>
<td>1</td>
<td>0.8</td>
</tr>
<tr>
<td>3</td>
<td>2.3</td>
</tr>
<tr>
<td>6</td>
<td>4.6</td>
</tr>
<tr>
<td>12</td>
<td>9.0</td>
</tr>
<tr>
<td>29</td>
<td>22</td>
</tr>
<tr>
<td>58</td>
<td>46</td>
</tr>
<tr>
<td>116</td>
<td>98</td>
</tr>
</tbody>
</table>
- Most negative pore pressure is measured from:
- Highly plastic fine-grained soil
土壤中空气存在的四种状态：(A) 空气完全连续阶段，(B) 空气部分连续阶段，(C) 空气内部连续阶段，(D) 空气完全封闭阶段
土壤中可能存在之饱和状态：(a) 钟摆状饱和，(b) 线状饱和，(c) 岛状饱和
Effective stress

- Terzaghi (1923):

\[ \bar{\sigma} = \sigma - u \]
\[ \bar{\sigma} = \sigma - ku \]

\( k = (1 - a \tan \varphi / \tan \phi) \) – for shear strength
\( k = 1 - C_s / C \) – for compressibility

\( a \): contact area of solids on a cross section
\( \varphi \): friction angle of solids
\( \phi \): friction angle of soil
\( C_s \): compressibility of solids
\( C \): compressibility of soil skeleton
Stress/Pressure

- $u_a$ – pore air pressure
- $u_w$ – pore water pressure
- $u_c = u_a - u_w$
Partly saturated soil

- Bishop’s “chi” theory (1955)

\[
\bar{\sigma} = (\sigma - u_a) + \chi (u_a - u_w)
\]

<table>
<thead>
<tr>
<th>Oven-dried soil</th>
<th>Saturated soil</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_r = 0%$</td>
<td>$S_r = 100%$</td>
</tr>
<tr>
<td>$\bar{\sigma} = \sigma - u_a$</td>
<td>$\bar{\sigma} = \sigma - u_w$</td>
</tr>
<tr>
<td>$\chi = 0$</td>
<td>$\chi = 0$</td>
</tr>
</tbody>
</table>
Bishop(1960) and Aitchison(1960)

\[
\overline{\sigma} = \sigma - k_1 u_w - k_2 u_a
\]

\[k_2 = 1 - k_1, \text{ if } k_1 = \chi \rightarrow \overline{\sigma} = (\sigma - u_a) + \chi(u_a - u_w)\]

\(\chi\) depends on \(S_r\) and void ratio \((e)\)
Fredlund & Morgenstern (1977)

- Control $s$, $u_a$, $u_w$ and observe the volume change of soils (fix two and let one of them change at a time).
- If any one of them were kept constant, volume remained the same.
Mongiovi及Tarantino

- Keep the volume of the soil specimen constant → observe the change of stress variables
- Very small changes of the stress variables
- The effective stress of unsaturated soil is controlled by two stress variables: \((\sigma - u_a)\) and \((u_a - u_w)\)
\[ \bar{\sigma} = (\sigma - u_a) + \chi(u_a - u_w) + \gamma T - \pi \]

\( \gamma = \) perimeter of referenced air-water interface
\( T = \) air-water interfacial tension \( \Rightarrow \) make solids get closer to each other
\( \pi = \) tension of dissolved material
飽和度Sr與結構張力、收縮表面張力 T 变化图
Allam & Sridharan (1987)

- Kaolinite
- $S_r = 0.3$, $\gamma T$ accounts for 13 – 36% of the suction
- Effect of $\pi$ was not significant
Fredlund (1979)

- Regina clay:
- Water content: 24% → 30%
- $\pi$ accounts for 40% → 80% of the total suction
Tensor expression

\[
\begin{bmatrix}
\sigma_x - u_a & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_y - u_a & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_z - u_a
\end{bmatrix}
\]

\[
\begin{bmatrix}
u_a - u_w & 0 & 0 \\
0 & u_a - u_w & 0 \\
0 & 0 & u_a - u_w
\end{bmatrix}
\]
When $S_r \rightarrow 100\%$, $u_w \rightarrow u_a$, $(u_a - u_w) \rightarrow 0$

\[
\begin{bmatrix}
\sigma_x - u_W & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_y - u_W & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_z - u_W
\end{bmatrix}
\]
Pore pressure coefficient $\chi$

- Bishop, Blight, and Donald (1960): $\chi$ is between 0 and 1
- Sparks (1963) based on force equilibrium:

$$\chi = \frac{\pi}{4} \left\{ \frac{\sin \theta + \cos \theta - 1}{\cos \theta} \right\}^2 - \frac{\pi T}{2 r p} \left\{ \frac{\sin \theta + \cos \theta - 1}{\cos \theta} \right\}$$
p = pore water pressure = \( \frac{T \cos \theta}{r} \left\{ \frac{1}{1 - \cos \theta} - \frac{1}{\sin \theta + \cos \theta - 1} \right\} \)

\( T \) = surface tension of water

\( r \) = radius of soil grains

\( \theta \) = angle between surface of water and center of soil grains,

When \( S_r \rightarrow 100\% \) and \( q \) is 45° \( \rightarrow \chi = 1.57 \)
Blight(1967)

- Get \( \chi \) by CD and CU tests
- When close to saturation, \( c \) may become \( > 1 \)
- Blight thought it might due to lack of lab data, and this was not real
  - However, he did not consider the effect of air-water interface
- If consider \( \gamma T \) in the formulation \( \rightarrow c = S_r \); but not experimental evidence to support this

\[
\chi = \frac{\frac{1}{2}(\sigma_1' + \sigma_3') - \frac{1}{2}(\sigma_1 + \sigma_3) - u_a}{(u_a - u_w)}
\]
Shear strength of unsaturated soil

- According to Bishop, the generalized Mohr-Coulomb failure criteria is:

\[
\tau_f = \bar{c} + (\sigma_n - u_a)\tan \phi + [\kappa(u_a - u_w)]\tan \phi
\]

- This equation is difficult to verify, because it is difficult to determine \( \kappa \)
Fredlund

- Suggests that the shear strength is controlled by two independent variables: \((\sigma - u_a)\) and \((u_a - u_w)\)
- He and Morgenstern suggest:

\[
\tau_f = \bar{c} + (\sigma_n - u_a) \tan \phi + (u_a - u_w) \tan \phi_b
\]

\((\sigma_n - u_a)\) and \((u_a - u_w)\) is the normal stress and matric suction on the failure surface.
\(\phi_b\) is the friction angle associated with the matric suction; generally it is not a constant, but decreases as the matric tension increases.
Relationship between $\phi_b$ and matric suction (Gan and Fredlund, 1988)
Lu (1992)

- Suggests the following criteria for expansive soils:

\[ \tau_f = \bar{c} + (\sigma_n - u_a) \tan \phi + P_s \tan \phi \]

\( P_s \): expansive pressure of soil
Xu (1998)

- Based on the concept of fractal dimension on expansive soil:

\[ \tau_f = \bar{c} + (\sigma_n - u_a)\tan \phi + k^n (u_a - u_w)^{mn+1} \tan \phi \]

- \( k \): fitting coefficient
- \( m, n \): fractal dimension parameters
The introduced theories and criteria have a common problem: difficult to measure the parameters.

Vanapalli And Fredlund (1996) developed an empirical equation relating the shear strength and the soil water characteristic curve.
Fredlund et al. (1978)

- The shear strength provided by the matric suction, $\tau_{us}$, is:

$$\tau_{us} = (u_a - u_w) \tan \phi_b$$
\( \phi_b \) varies with the water content of soil, i.e., it varies with the matric suction of soil →

\[
\tau_{us} = (u_a - u_w) \left[ (\Theta^k \tan \phi) \right]
\]

\[
\tan \phi_b = \frac{d\tau}{d(u_a - u_w)} = \left[ (\Theta)^k + (u_a - u_w) \frac{d(\Theta)^k}{d(u_a - u_w)} \right] \tan \bar{\phi}
\]

\( \Theta \) is the normalize volumetric water content
\( \theta_s \) is the volumetric water content at saturation
\( k \) is a fitting parameter → has to be determined by experiments

\[
\Theta = \frac{\theta}{\theta_s}
\]
Vanapalli et al. (1996)

- Obtain shear strength based on SWCC

\[ \tau_f = \bar{c} + (\sigma_n - u_a)\tan \bar{\phi} + (u_a - u_w)\left(\frac{\theta - \theta_r}{\theta_s - \theta_r}\right)\tan \bar{\phi} \]